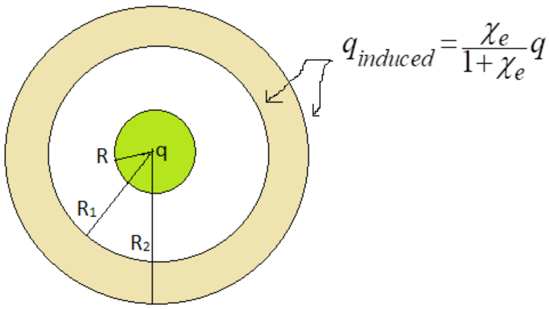
# *Homework 6 (Solutions): Electric Fields in Matter*

**Problem 1.** Say we have a spherical shell surrounding a charge q of radius R. Derive the following equation for the induced charge on the inner/outer surfaces of the shell.



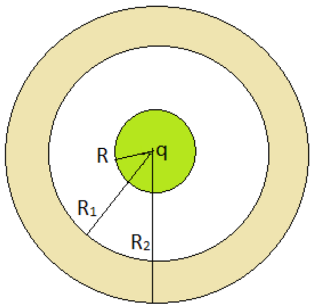

So the charge density on each surface is:

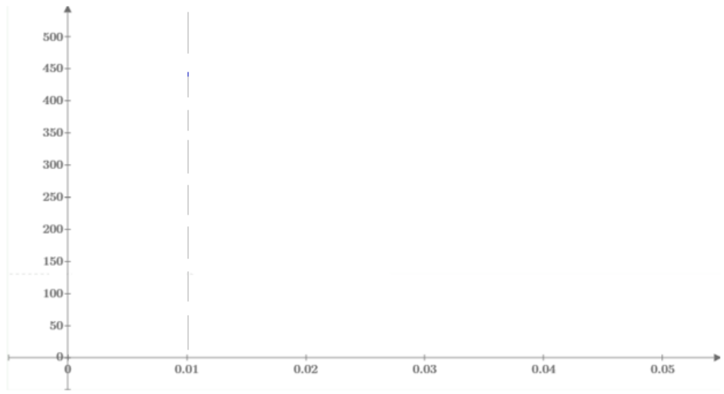
$$\begin{aligned}\sigma &= \chi_e \varepsilon_0 E \\ &= \chi_e \varepsilon_0 \frac{E_0}{\kappa_e} \\ &= \frac{\chi_e}{1+\chi_e} \varepsilon_0 \frac{kq}{r^2}\end{aligned}$$

And therefore the total charge is:

$$\begin{aligned}q_{induced} &= \sigma A \\ &= \left( \frac{\chi_e}{1+\chi_e} \varepsilon_0 \frac{kq}{r^2} \right) 4\pi r^2 \\ &= \frac{\chi_e}{1+\chi_e} 4\pi \varepsilon_0 kq \\ &= \frac{\chi_e}{1+\chi_e} q\end{aligned}$$

**Problem 2.** Now with reference to the previous setup, suppose q = 5pC charge with a radius R = 1cm. Let $R_1$ = 2cm, and $R_2$ = 3cm. The empty space white space in between and outside is just air. If the shell is a perfect insulator ($\chi_e$ = 0), answer the following questions:

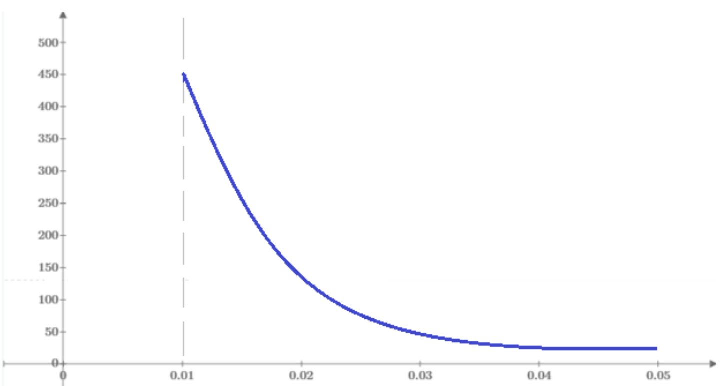



(a) Plot the electric field as a function of r, from r = 1cm to 5cm.
So the external field is:

$$E_0 = \frac{kq_{enclosed}}{r^2} = \frac{(9\times10^{9})(5\times10^{-12})}{r^2} = \frac{0.045}{r^2}$$

This would be the field in the air. In the insulator ($\kappa_e = 1 + \chi_e = 1 + 0 = 1$) it would be given by $E = E_0/\kappa_e$ = same as $E_0$.




(b) Determine the induced charges on the surfaces, and specify their signs.
So the charge density on each surface is:

$$q_{induced} = \frac{\chi_e}{1+\chi_e} q = \frac{0}{1+0} \cdot 5\text{pC} = 0$$

(c) Calculate the electric potential energy, ‘elastic’ potential energy, and total potential energy stored in the shell.
Energies are given by:

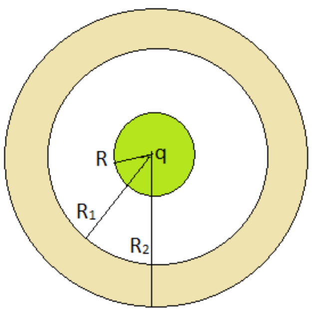
$$PE_E = \int_{R_1}^{R_2} \frac{\varepsilon_0}{2} E^2 dV_{sphere} = \int_{R_1}^{R_2} \frac{\varepsilon_0}{2}\left(\frac{0.045}{r^2}\right)^2 \cdot 4\pi r^2 dr = 2\pi\varepsilon_0 \cdot 0.045^2 \int_{0.02}^{0.03} \frac{dr}{r^2} = 1.88\,\text{pJ}$$

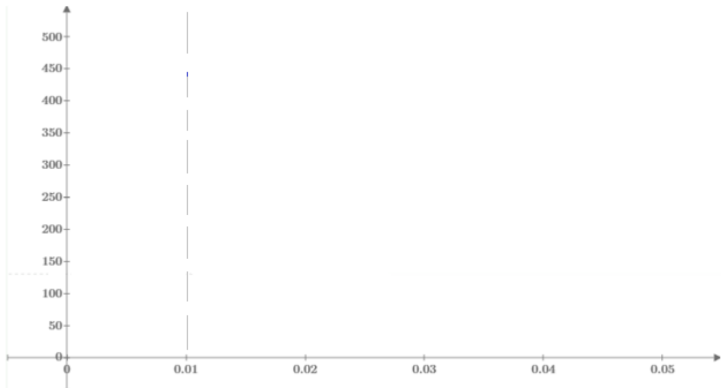
$$PE_{elastic} = \int_{R_1}^{R_2} \chi_e \frac{\varepsilon_0}{2} E^2 dV_{sphere} = \chi_e PE_E = 0 \cdot PE_E = 0$$

$$PE_{total} = PE_E + PE_{elastic} = 1.88\,\text{pJ}$$

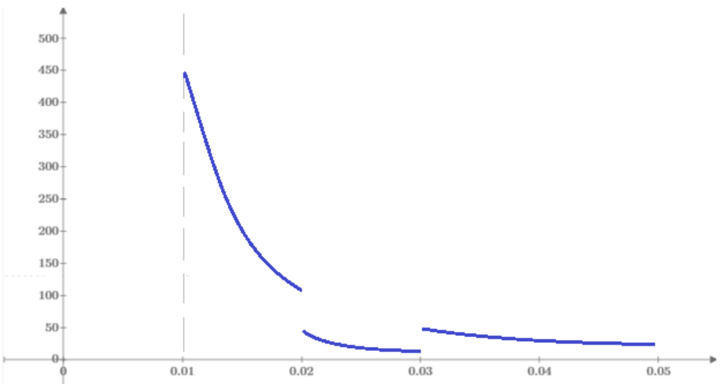
(d) What would be the dielectric strength of a perfect insulator?
$E = \infty$.

**Problem 3.** Now let's fill the shell with pyrex ($\kappa_e$ = 5.6).




(a) Plot the electric field as a function of r, from r = 1cm to 5cm.
The field in air would still be given by $E_0 = 0.045/r^2$, but in the pyrex ($\kappa_e$ = 5.6) by $E = E_0/\kappa_e$. Plotted above.




(b) Determine the induced charges on the surfaces, and specify their signs.
So the charge density on each surface is:

$$q_{induced} = \frac{\chi_e}{1+\chi_e} q \qquad \chi_e = \kappa_e - 1$$

$$= \frac{4.6}{1+4.6}(5\text{pC})$$

$$= 4.11\text{pC}$$

So this would be -4.11pC on the inner surface $R_1$, and +4.11pC on the outer surface $R_2$.

(c) Calculate the electric potential energy, 'elastic' potential energy, and total potential energy stored in the shell.

Energies are given by:

$$PE_E = \int_{R_1}^{R_2} \frac{\varepsilon_0}{2} E^2 dV_{sphere} = \int_{R_1}^{R_2} \frac{\varepsilon_0}{2}\left(\frac{E_0}{\kappa_e}\right)^2 dV_{sphere} = \frac{1}{\kappa_e^2}\int_{R_1}^{R_2} \frac{\varepsilon_0}{2} E_0^2 dV_{sphere} = \frac{1.88\text{pJ}}{5.6^2} = 0.06\,\text{pJ}$$

$$PE_{elastic} = \int_{R_1}^{R_2} \chi_e \frac{\varepsilon_0}{2} E^2 dV_{sphere} = \chi_e PE_E = 5.46 \cdot PE_E = 0.34\text{pJ}$$

$$PE_{total} = PE_E + PE_{elastic} = 0.40\,\text{pJ}$$

(d) The dielectric strength of pyrex is 14MN/C. What would our center q have to be to induce dielectric breakdown?

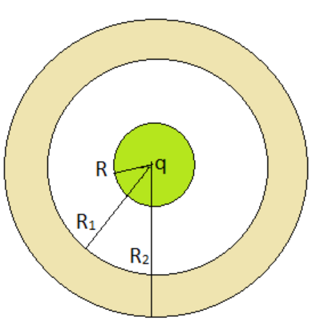
Well, the field would be strongest at the closest edge. So we need:

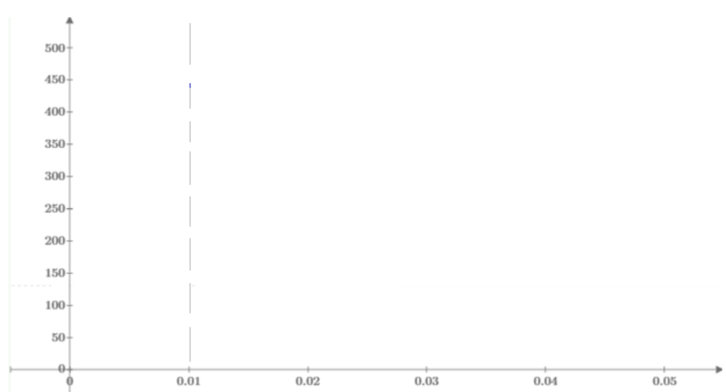
$$E(r = R_1) = 14\times10^6$$

$$\frac{1}{\kappa_e}\frac{kq}{R_1^2} = 14\times10^6$$

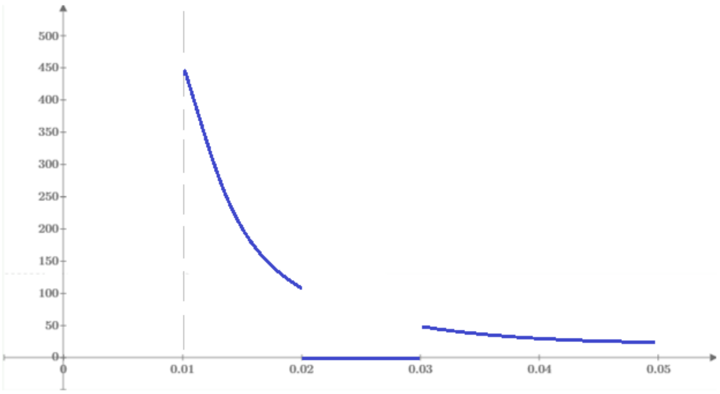
$$q = \frac{14\times10^6}{k}\kappa_e R_1^2 = \frac{14\times10^6}{9\times10^9}(5.6)(0.02)^2 = 3.5\,\mu\text{C}$$

**Problem 4.** Now let's make the shell a perfect metal ($\kappa_e = \infty$).



(a) Plot the electric field as a function of r, from r = 1cm to 5cm.
Field in air is still $E_0 = 0.045/r^2$, but in metal ($\kappa_e = \infty$) by $E = E_0/\infty = 0$. Plotted above.




(b) Determine the induced charges on the surfaces, and specify their signs.
Induced charge is:

$$
\begin{aligned}
q_{induced} &= \frac{\chi_e}{1+\chi_e} q \\
&= \frac{\infty}{1+\infty}(5\,\text{pC}) \\
&= 5\text{pC}
\end{aligned}
$$

So this would be -5pC on the inner surface $R_1$, and +5pC on the outer surface $R_2$.

(c) Calculate the electric potential energy, 'elastic' potential energy, and total potential energy stored in the shell.
And last,

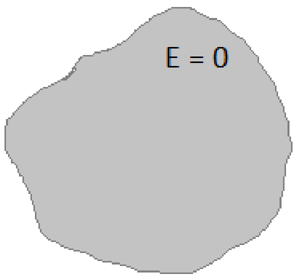
$$PE_E = \int_{R_1}^{R_2} \frac{\varepsilon_0}{2} E^2 dV_{sphere} = \int_{R_1}^{R_2} \frac{\varepsilon_0}{2} \cdot 0^2\, dV_{sphere} = 0\,\text{pJ}$$

$$PE_{elastic} = \int_{R_1}^{R_2} \chi_e \frac{\varepsilon_0}{2} E^2 dV_{sphere} = \chi_e PE_E = 0\,\text{pJ}$$

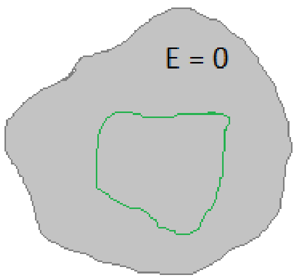
$$PE_{total} = PE_E + PE_{elastic} = 0\,\text{pJ}$$

(d) What is the dielectric strength of a metal?
E = anything greater than 0 basically.

**Problem 5.** Use Gauss's law to show that as long as E = 0 inside a substance, like a metal for instance, there can be no *net* charge *within* that surface. I suggest you draw a Gaussian surface within the substance and see what happens, and your proof should take about three lines max.

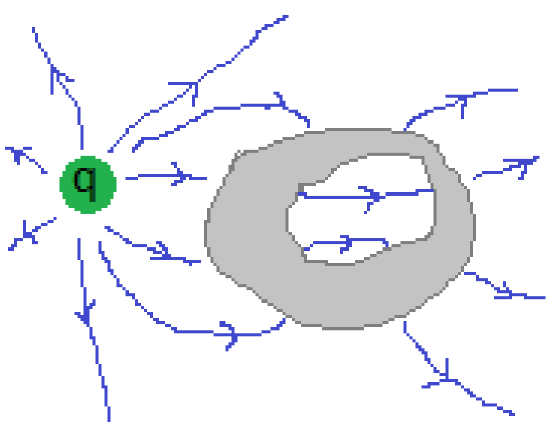



So, drawing a Gaussian surface within the substance,

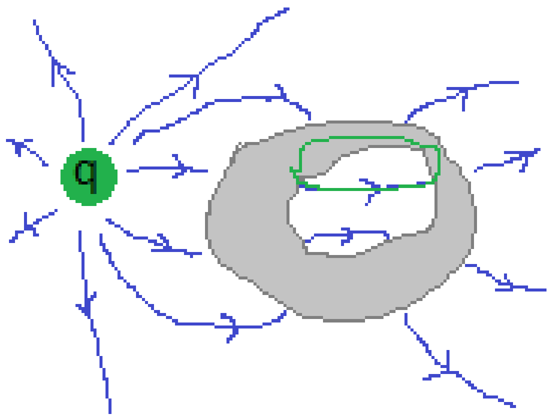



Since E = 0, then $\int E \cdot dA = 0 \longrightarrow q_{enclosed} = 0 \longrightarrow$ can be no net charge within.

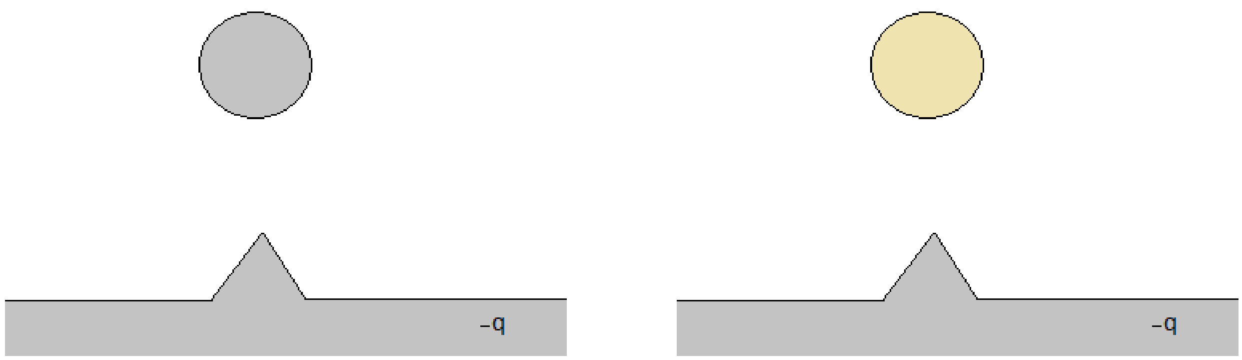
**Problem 6.** Like my drawing? In problem 4, you should've seen that while E = 0 inside the metal, it still leaked through the metal into the air outside. But now consider this situation. Say we have a charge outside the metal. Can the field lines leak into the cavity *inside* the metal? Prove your answer one way or the other by considering the consequences of the fact that ΔV must be 0 around a closed loop. Note, this is the reason cell phone signals get lost in elevators, metal (Faraday) cages are used to shield sensitive electronics, and people wear tin foil hats to protect their brains from aliens' presumably EM probes.




So, consider the green loop. $\Delta V = \Delta V_{\text{along field line}} + \Delta V_{\text{in metal}} = -E\ell + 0$. But since $\Delta V = 0$ by necessity, we must have $E = 0$.




**Problem 7**. Consider the following setups. Charge -q is deposited on a metal. And a neutral sphere hovers magically above it. Draw where the -q charges will distribute themselves in the metal, and where the induced charge, if any, will distribute itself in the sphere. Then draw a rough picture of what you expect the electric field lines to look like. On the left, assume the magic sphere is a perfect metal, and on the right assume the magic sphere is a perfect insulator.




So should look like this. On left, all charges, induced or otherwise must congregate on surfaces, and most will be at the sharp edge. There is no field inside either metal, and all field lines must hit the metal perpendicular to the surface. On the right we have some, but no induced charge since it's a perfect insulator, and therefore no E field distortion at all in its vicinity.